J. Fluid Mech. (1979), *vol.* 90, *part* 4, *pp.* 781–793
Printed in Great Britain

The drag on a sphere moving axially in a long rotating container

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(Received 30 June 1978)

A container of viscous incompressible liquid is bounded by rigid parallel planes and rotates steadily about an axis normal to these planes. A rigid sphere moves steadily parallel to the rotation axis and the Rossby and Ekman numbers characterizing the motion are both small. The drag on the sphere is calculated in the case when the length of the Taylor column is comparable to the axial dimension of the container. Viscous effects are allowed for in the boundary of the Taylor column, but the Ekman layers on the sphere and on the bounding planes are shown not to affect the drag to leading order. The determination of the drag involves solving dual integral equations. This is done numerically and, for the limiting cases of long and short containers, analytically. The interaction of the Taylor column and the ends of the container leads to an increase in the drag over its value in an unbounded fluid, but the increase is smaller than that measured by Maxworthy (1970).

1. Previous work and objectives

The motion induced by a sphere rising parallel to the axis of a rotating liquid has been the subject of several theoretical and experimental investigations. These have had as their objective the addition of quantitative detail to the qualitative experimental and theoretical picture of such flows given in a famous series of papers by Taylor (1917, 1921, 1922).

The most interesting case is when the Rossby number R_0, defined by

$$R_0 = U/(2a\Omega), \tag{1.1}$$

is small; here U is the velocity of the sphere, $2a$ is its diameter and Ω is the angular velocity of the liquid. In this case Taylor showed that the sphere carried with it a long column of fluid whose axis is parallel to the rotation axis. However, the steady, linearized, inviscid equations used by Taylor possess an infinite number of solutions all satisfying the boundary conditions on the sphere and all having a columnar structure.

The first attempt to remove this indeterminacy is due to Grace (1926), who solved an initial-value problem for the inviscid equations

$$\partial \mathbf{u}/\partial t + 2\mathbf{\Omega} \times \mathbf{u} = -\rho^{-1}\nabla p \tag{1.2}$$

and

$$\operatorname{div} \mathbf{u} = 0, \tag{1.3}$$

where $\mathbf{u}$ is the liquid velocity, ρ its density and p the reduced pressure. Grace was not able to solve the problem completely, but his analysis suggested that the drag on an impulsively started sphere settled down to a steady value in a time comparable to the rotation period. A complete discussion was provided by Stewartson (1952), who showed that the drag attained the asymptotic value D_0, where

$$D_0 = \tfrac{16}{3}\rho\Omega U a^3. \tag{1.4}$$

The problem can be looked at as one of inertial-wave generation and propagation, so that a steady drag is achieved only after the transients have propagated away from the vicinity of the sphere. This aspect of the problem was studied by Bretherton (1967), who worked out the details for the two-dimensional analogue in which the sphere is replaced by a circular cylinder.

The neglect of viscosity can be legitimate only if the Ekman number E, defined by

$$E = \nu/(a^2\Omega), \tag{1.5}$$

is sufficiently small. Here ν is the kinematic viscosity. Since, as Stewartson showed, singularities develop on the Taylor-column boundary as $t \to \infty$, the neglect of viscosity cannot, even then, be uniformly valid in the flow field.

This suggests an alternative approach, which retains the viscous terms in the governing equations, but treats the motion as steady. Thus (1.1) is replaced by

$$2\mathbf{\Omega} \times \mathbf{u} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{u}. \tag{1.6}$$

This approach derives from the work of Morrison & Morgan (1956) and W. S. Childress (unpublished) and is discussed by Moore & Saffman (1969, hereafter referred to as I).

The Taylor column has a long but finite length $O(aE^{-1})$ and within the Taylor column the swirl velocity is $O(U)$. Thus the Ekman suction velocity, which is $O(UE^{\frac{1}{2}})$, is negligible and the problem can be discussed in terms of an approximation to (1.6) in which axial derivatives are neglected. To leading order, the drag arises from the pressure difference between the fore and aft Taylor columns and proves to agree with Stewartson's result (1.4).

The problem was studied experimentally by Maxworthy (1970), who released small buoyant spheres of diameters $\frac{1}{2}$ in. and $\frac{3}{4}$ in. in a rotating tank of water of depth 5 ft, the dimensions being chosen to make the effects of the ends of the tank as small as feasible. For small values of R_0 Maxworthy found that the drag was higher than that given by Stewartson by a factor of nearly two, even after an empirical end-wall correction had been included. This discrepancy must cast doubt on the applicability of the linearized theories and its importance has been stressed recently by Barnard & Pritchard (1975), who provide a valuable set of comparisons between linearized theory and experiment.

Examination of Maxworthy's results shows that the parameter defined by

$$\delta = hE/2a, \tag{1.7}$$

which is the ratio of the depth h of the cylinder to the Taylor-column height $2aE^{-1}$, is actually as small as 0·2 for the data on which the discrepancy is based.† This suggests

† A precise definition of the length of the Taylor column is the largest distance from the sphere at which the liquid velocity on the axis is equal to the speed of the body. Barnard & Pritchard (1975), using the results of I, obtain the value $5{\cdot}26 \times 10^{-2}aE^{-1}$, which is in good agreement with the value $5{\cdot}88 \times 10^{-2}aE^{-1}$ derived from Maxworthy's experiments.

that it would be of interest to calculate the effect of end walls on the drag, to see if they could have an effect comparable to that observed by Maxworthy.

We attempt this calculation in the present paper. The calculation of I is repeated with rigid planes at right angles to the axis of rotation. Axial derivatives are again dropped from the governing equation (1.6) and the Ekman layers on the end walls and sphere are neglected. Thus the upper end wall could equally be a free surface, as far as our theory is concerned. The neglect of Ekman suction restricts the analysis to containers for which $h \gg aE^{-\frac{1}{2}}$ and there is thus no overlap with the analysis of Moore & Saffman (1968) for $h \ll aE^{-\frac{1}{2}}$. We neglect the effect of the sides of the container and treat the liquid as radially unbounded. The dual integral equations treated in I are replaced by a more complicated pair which have, in general, to be solved numerically.

The dual integral equations are derived in § 2 and Tranter's method (1971, p. 111) is used to convert them to an algebraic system in § 3. Approximate solutions are possible when $\delta \gg 1$ and when $\delta \ll 1$ and these limits are explored in §§ 4 and 5 respectively.

Finally, in § 6, we look at the comparison with Maxworthy's results.

2. The derivation of the dual integral equations

In this section we derive the equations and boundary conditions determining the motion and show how they reduce to dual integral equations.

We use cylindrical polar co-ordinates (r, θ, z) whose origin is at the centre of the sphere and whose axis is parallel to the axis of rotation. The fluid is bounded by rigid infinite flat plates at $z = h_T$ and $z = -h_B$, where $h_T + h_B = h$, the length of the container.† The plates rotate with angular velocity Ω so that the undisturbed state of the fluid is one of rigid rotation.

Outside Ekman layers on the sphere and end plates, z derivatives are unimportant, so that the equations of motion (1.6) can be approximated by

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.1}$$

$$2\Omega u = \nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right), \tag{2.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{r\partial r}\right), \tag{2.3}$$

while continuity requires

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0. \tag{2.4}$$

Here (u, v, w) are the velocity components in the (r, θ, z) directions. We are able to use steady equations even though the flow field changes as h_T and h_B vary because the sphere moves for $O(R_0 a/h)$ rotation periods before a significant change of configuration occurs. The neglect of the viscous force in the radial momentum balance (2.1) was justified in I and the argument need not be repeated here. The full r dependence

† As in I, we denote $z > 0$ by T and $z < 0$ by B.

of the viscous force in (2.2) and (2.3) is retained because the Stewartson layer at the Taylor-column boundary can be of thickness comparable to the sphere radius a.

If it turns out that the solution has a swirl velocity $O(U)$ then the Ekman suction velocity will be $O(UE^{\frac{1}{2}})$ and can be neglected in formulating the boundary conditions which become

$$w = 0 \quad \text{on} \quad z = h_T \quad (0 \leqslant r < \infty), \tag{2.5}$$

$$w = 0 \quad \text{on} \quad z = -h_B \quad (0 \leqslant r < \infty) \tag{2.6}$$

and

$$\left.\begin{aligned} &w = U \quad \text{on} \quad z = \pm 0 \quad (0 \leqslant r \leqslant a), \\ &u, v, w \text{ continuous across } z = 0 \quad (a \leqslant r < \infty). \end{aligned}\right\} \tag{2.7}$$

We have transferred the boundary conditions on the surface of the sphere to $z = 0$ because the velocity field does not vary significantly with z on the scale of the sphere. Note that we have not specified the angular velocity of the sphere, because an angular velocity of order ΩR_0 relative to the rotating frame would produce an Ekman suction of order $UE^{\frac{1}{2}}$ and we are neglecting velocities of this order of magnitude. In Maxworthy's experiments the angular velocity would be determined by the condition that the sphere was subject to zero torque.

It can easily be verified that

$$v(r,z) = \int_0^\infty J_1(kr)\{\alpha(k)\exp(\tfrac{1}{2}\nu\Omega^{-1}k^3z) + \beta(k)\exp(-\tfrac{1}{2}\nu\Omega^{-1}k^3z)\}\,dk \tag{2.8}$$

and
$$w(r,z) = \int_0^\infty J_0(kr)\{\alpha(k)\exp(\tfrac{1}{2}\nu\Omega^{-1}k^3z) - \beta(k)\exp(-\tfrac{1}{2}\nu\Omega^{-1}k^3z)\}\,dk \tag{2.9}$$

are solutions of the equations of motion (2.1)–(2.4) for arbitrary weighting functions $\alpha(k)$ and $\beta(k)$. However, the velocity field is not analytic across $z = 0$ because of the presence of the sphere so we must use weighting functions $\alpha_T(k)$ and $\beta_T(k)$ for $z > 0$ and $\alpha_B(k)$ and $\beta_B(k)$ for $z < 0$.

The problem is thus to calculate the four weighting functions. Since the boundary conditions (2.5) and (2.6) hold for all r, we have immediately

$$\alpha_T(k)\exp(\tfrac{1}{2}\nu\Omega^{-1}k^3h_T) - \beta_T(k)\exp(-\tfrac{1}{2}\nu\Omega^{-1}k^3h_T) = 0 \tag{2.10}$$

and

$$\alpha_B(k)\exp(-\tfrac{1}{2}\nu\Omega^{-1}k^3h_B) - \beta_B(k)\exp(\tfrac{1}{2}\nu\Omega^{-1}k^3h_B) = 0. \tag{2.11}$$

Moreover, w is continuous across $z = 0$ for all r, so that

$$\alpha_T(k) - \beta_T(k) = \alpha_B(k) - \beta_B(k). \tag{2.12}$$

We can use (2.10)–(2.12) to reduce the number of unknown functions to one.

A quantity of physical interest is the difference in swirl velocity between the upper and lower surfaces of the sphere, because the drag force can be expressed in terms of this difference. Now

$$v(r,+0) - v(r,-0) = \int_0^\infty J_1(kr)\{\alpha_T - \alpha_B + \beta_T - \beta_B\}\,dk$$

and this suggests that we work with a new unknown $\gamma(k)$ defined by

$$\gamma(k) = \alpha_T(k) - \alpha_B(k) + \beta_T(k) - \beta_B(k). \tag{2.13}$$

We now find that the conditions on $z = 0$ lead to

$$\int_0^\infty \gamma(k) J_0(kr) \frac{[1-\exp(-\nu\Omega^{-1}h_T k^3)][1-\exp(-\nu\Omega^{-1}h_B k^3)]}{1-\exp(-\nu\Omega^{-1}hk^3)} dk = -2U \quad (0 \leqslant r \leqslant a)$$

and

$$\int_0^\infty \gamma(k) J_1(kr)\, dk = 0 \quad (r \geqslant a).$$

If we use the identity

$$d[rJ_1(kr)]/dr = krJ_0(kr),$$

we can change the Bessel function in the first equation to one of order unity. Then, if we introduce dimensionless variables through the equations

$$u = ka, \quad -2aUA(u) = \gamma(k), \quad s = ra^{-1}, \tag{2.14}$$

we get

$$\int_0^\infty u^{-1}A(u)\,\Delta(u)\,J_1(us)\,du = \tfrac{1}{2}s \quad (0 \leqslant s \leqslant 1), \tag{2.15}$$

and

$$\int_0^\infty A(u)\,J_1(us)\,du = 0 \quad (1 \leqslant s), \tag{2.16}$$

where

$$\Delta(u) = \frac{[1-\exp(-\delta_T u^3)][1-\exp(-\delta_B u^3)]}{1-\exp[-(\delta_T+\delta_B)u^3]} \tag{2.17}$$

and where $\delta_T = Eh_T/a$ and $\delta_B = Eh_B/a$. Thus δ_T and δ_B are $O(1)$ in the case considered.

It can be shown that the drag D on the sphere is given by

$$D = -2\Omega\pi\rho \int_0^a r^2\{v(r,0+)-v(r,0-)\}\,dr,$$

(Moore & Saffman 1968), so that

$$D = 4\pi\Omega\rho a^3 U \int_0^1 s^2 \int_0^\infty J_1(us)\,A(u)\,du\,ds. \tag{2.18}$$

It is worth noting that $\Delta(u)$ is symmetric in δ_T and δ_B, so that $A(u)$ and hence D have the same symmetry. Thus the drag correction due to the end plates is an even function of the distance of the sphere from the mid-point of the apparatus. This is a consequence of the linearity of the governing equations: the effect of an end wall cannot depend on whether the sphere is approaching or receding. In future (and without loss of generality) we assume $\delta_T \leqslant \delta_B$.

In § 3 the solution of (2.15) and (2.16) is reduced to an infinite system of algebraic equations by Tranter's method (1971, p. 111).

3. Reduction to algebraic equations

The key step in Tranter's method is the recognition that the second equation of the pair, equation (2.16), is satisfied for all a_m by the series

$$A(u) = u \sum_{m=0}^{\infty} a_m j_{2m+1}(u), \tag{3.1}$$

where j_n denotes the spherical Bessel function of order n. This follows from the result (Watson 1958, p. 404)

$$\int_0^\infty u j_{2m+1}(u)\,J_1(us)\,du = \begin{cases} \dfrac{\pi^{\frac{1}{2}}(m+1)!}{\Gamma(m+\frac{1}{2})}\, s(1-s^2)^{-\frac{1}{2}} F_m(\tfrac{3}{2}, 2, s^2) \quad (s < 1), & (3.2a) \\ 0 \quad (s > 1), & (3.2b) \end{cases}$$

where F_m denotes the Jacobi polynomial. Thus the problem is to determine the coefficients a_m such that the first equation of the pair, equation (2.15), is satisfied.

If we substitute the expansion (3.1) into (2.15) and invert the order of summation and integration, we find

$$\sum_0^\infty a_m \int_0^\infty \Delta(u) j_{2m+1}(u) J_1(us)\,du = \tfrac{1}{2}s \quad (0 \leqslant s \leqslant 1). \tag{3.3}$$

The problem is to convert this equation into an infinite system of algebraic equations for the a_m and Tranter showed how this could be accomplished. He pointed out that (3.2) furnishes the Hankel transform of j_{2m+1}, so that inverting (3.2) gives

$$j_{2m+1}(u) = \frac{\pi^{\frac{1}{2}}(m+1)!}{\Gamma(m+\frac{1}{2})} \int_0^1 s^2(1-s^2)^{-\frac{1}{2}} F_m(\tfrac{3}{2}, 2, s^2) J_1(us)\,ds. \tag{3.4}$$

So if we multiply (3.3) by the function $s^2(1-s^2)^{-\frac{1}{2}}F_n(\frac{3}{2}, 2, s^2)$ and integrate from 0 to 1 we shall get

$$\sum_0^\infty a_m \int_0^\infty \Delta(u) j_{2m+1}(u) j_{2n+1}(u) \frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}(n+1)!}\,du = \int_0^1 \tfrac{1}{2}s^3(1-s^2)^{-\frac{1}{2}} F_n(\tfrac{3}{2}, 2, s^2)\,ds. \tag{3.5}$$

Use of the orthogonality relations satisfied by Jacobi polynomials (Magnus & Oberhettinger 1954, p. 83) shows that the right-hand side of (3.5) vanishes unless $n = 0$, when it has the value $\frac{1}{3}$. Thus the infinite system can be written

$$\sum_{m=0}^\infty a_m \int_0^\infty \Delta(u) j_{2m+1}(u) j_{2n+1}(u)\,du = \tfrac{1}{3}\delta_{0,n}, \tag{3.6}$$

where $\delta_{p,q}$ is the Kronecker delta.

The integral will be evaluated numerically. However, $\Delta(u) \to 1$ and $j_{2m+1}\, j_{2n+1}$ is $O(u^{-2})$ as $u \to \infty$, so that convergence would be slow at the upper limit. To improve the convergence, we note that

$$\Delta(u) - 1 \sim \exp[-(\delta_T + \delta_B)u^3] - \exp(-\delta_T u^3) - \exp(-\delta_B u^3)$$

as $u \to \infty$, while

$$\int_0^\infty j_{2m+1}(u) j_{2n+1}(u)\,du = \frac{\pi\delta_{m,n}}{2(4n+3)}. \tag{3.7}$$

Thus the infinite system of algebraic equations can be rewritten in the form

$$\frac{\pi a_n}{2(4n+3)} + \sum_{m=0}^\infty a_m I_{m,n} = \tfrac{1}{3}\delta_{0,n} \quad (n = 0, 1, 2, \ldots), \tag{3.8}$$

where

$$I_{m,n} = \int_0^\infty (\Delta(u) - 1) j_{2m+1}(u) j_{2n+1}(u)\,du. \tag{3.9}$$

The integral defining $I_{m,n}$ is easily evaluated numerically.

Once the coefficients a_m have been found, the drag can be calculated and, in fact, use of some standard results shows that

$$D = \tfrac{8}{3}\pi\rho\Omega U a^3 a_0. \tag{3.10}$$

As a check on this result, we observe that as δ_T and δ_B become infinite, corresponding to the unbounded case, $\Delta(u) - 1$ tends to 0, for $u \neq 0$. Thus $I_{m,n} = 0$ for the unbounded case and all the off-diagonal terms in (3.8) vanish; the solution is simply $a_0 = 2/\pi$, $a_1 = a_2 = \ldots = 0$. Thus $D = \frac{16}{3}\rho\Omega U a^3$, in agreement with the result for an unbounded region.

For values of δ_T which are not small (recall that we have restricted ourselves to $\delta_T \leqslant \delta_B$) we can show that the off-diagonal elements $I_{m,n}$ tend rapidly to zero as either m or n or both tend to infinity. To see this we note that, in view of Poisson's integral (Abramowitz & Stegun 1964, p. 483),

$$|j_n(u)| \leqslant u^n\pi/(2^{n+1}n!),$$

if u is real and positive. Hence

$$|I_{m,n}| \leqslant \frac{\pi^2}{(2n+1)!\,(2m+1)!\,2^{2m+2n+3}}\int_0^\infty (1-\Delta(u))\,u^{2m+2n+2}du.$$

But if $\delta_T \leqslant \delta_B$, as we are assuming,

$$1-\Delta(u) \leqslant 2\exp(-\delta_T u^3),$$

so that
$$|I_{m,n}| \leqslant \frac{\pi^2\Gamma(\frac{2}{3}m+\frac{2}{3}n+1)}{3(2n+1)!\,(2m+1)!\,2^{2n+2m+2}\delta_T^{\frac{2}{3}m+\frac{2}{3}n+1}},$$

from which the smallness of the off-diagonal elements follows.

Since the off-diagonal elements decay so rapidly as we leave the main diagonal, we expect that replacing the infinite system (3.8) by an $N\times N$ system will produce an accurate solution for a_0 even for modest values of N. We examine this point in § 6. Furthermore, we can see that all the off-diagonal elements will be small if $\delta_T \gg 1$, so that an approximate solution of the infinite system can be found in this case. We determine this solution in § 4.

4. Approximate solution for a long container

In this section we examine the approximation which becomes possible when $\delta_T \gg 1$. This means that the parameter δ defined in (1.7) is large, so that the end walls have only a small effect on the flow.

The analysis is similar to that given by Tranter (1971, p. 119) for a parallel-disk condenser with a large gap. For simplicity, we consider first the case $\delta_T = \delta_B$ so that

$$\delta_T = \delta_B = \delta = hE/2a; \tag{4.1}$$

thus the sphere is midway between the end plates. The function $\Delta(u)-1$ now simplifies to

$$\Delta(u)-1 = -2/[1+\exp(\delta u^3)]. \tag{4.2}$$

If we substitute the integral representation

$$j_{2m+1}(x)j_{2n+1}(x) = x^{-1}\int_0^{\frac{1}{2}\pi} J_{2m+2n+3}(2x\cos\phi)\cos\{2(m-n)\phi\}\,d\phi$$

into the definition of $I_{m,n}$ we get, after expanding the Bessel function as a power series and carrying out the ϕ integration,

$$I_{m,n} = \frac{2\pi}{3}\sum_{s=0}^{\infty}\delta^{-\nu}A_{m,n,s}\int_0^\infty \frac{y^{\nu-1}}{1+e^y}\,dy, \tag{4.3}$$

where $\nu = \frac{2}{3}(m+n+s)+1$ and

$$A_{m,n,s} = \frac{(-1)^{s+1}(2m+2n+2s+3)!}{2^{2m+2n+2s+4}s!(2m+2n+s+3)!\,\Gamma(2m+s+\frac{5}{2})\,\Gamma(2n+s+\frac{5}{2})}. \tag{4.4}$$

The integral in (4.3) can be expressed as a Riemann zeta function so we get

$$I_{m,n} = \frac{2\pi}{3}\sum_{s=0}^{\infty}\frac{A_{m,n,s}}{\delta^{\nu}}\,\Gamma(\nu)\,\zeta(\nu)\,(1-2^{1-\nu}), \tag{4.5}$$

where $\zeta(\nu)\,(1-2^{1-\nu})$ must be replaced by its limiting value ln 2 when $\nu = 1$.

We have thus expressed $I_{m,n}$ as a power series in $\delta^{-\frac{2}{3}}$ and consequently we can find approximations to $I_{m,n}$ when $\delta \gg 1$. Moreover $I_{m,n}$ is $O(\delta^{-\frac{2}{3}(m+n)-1})$, so that the off-diagonal elements in the infinite system (3.8) are small and it can easily be solved approximately. We find that for $n \geqslant 1$

$$a_n = O(\delta^{-\frac{2}{3}n-1}) \tag{4.6}$$

and that

$$a_0(\tfrac{1}{6}\pi + I_{0,0}) = \tfrac{1}{3} + O(\delta^{-\frac{10}{3}}). \tag{4.7}$$

Thus, recalling (1.4) and (3.10), we have

$$\frac{D}{D_0} = \left(1 - \frac{0{\cdot}09806}{\delta} + \frac{0{\cdot}02092}{\delta^{\frac{5}{3}}} - \frac{0{\cdot}002459}{\delta^{\frac{7}{3}}} + \frac{0{\cdot}0002159}{\delta^{3}}\right)^{-1} + O(\delta^{-\frac{10}{3}}). \tag{4.8}$$

This result confirms that the drag is increased by the presence of the end walls, although the smallness of the coefficient of δ^{-1} shows that the effect is not as large as might have been anticipated.

Finally we return to the general case, $\delta_T \neq \delta_B$, to see how the drag correction is affected by the position of the sphere. If δ_T/δ_B is $O(1)$ we can evaluate $I_{0,0}$ to leading order and we find

$$\frac{D}{D_0} = 1 - \frac{1}{9\pi\delta}\left(\psi\left(\frac{\delta_T}{2\delta}\right) + \psi\left(\frac{\delta_B}{2\delta}\right) + 2C\right), \tag{4.9}$$

where ψ is the logarithmic derivative of the gamma function and C is Euler's constant. This result reveals that the effect of the position of the sphere is not large, the correction being increased by about 20 % when $\delta_T/\delta_B = 0{\cdot}5$ and by about 80 % when $\delta_T/\delta_B = 0{\cdot}25$. The drag is least in the symmetric position and in fact this proves to be true for all values of δ.

In the next section we consider the opposite limit in which δ_T and δ_B are small.

5. Approximate solution for a short container

By a 'short' container in the context of this problem we mean merely that $h \ll aE^{-1}$. In fact it will emerge that we must have $h \gg aE^{-\frac{1}{2}}$ in order that the Ekman layers can be neglected, so that $h \gg a$ still.

Examination of (4.4), which holds for $\delta_T = \delta_B$ $(=\delta)$, reveals that, for $\delta \ll 1$, the off-diagonal elements in the infinite system (3.8) becomes small only at large values of m and n. This behaviour can be shown to persist for $\delta_T < \delta_B$, so that the truncated system has to be made large when δ is small. Thus we must precede differently in this case.

If we examine the definition of $\Delta(u)$ given by (2.17), we can see that, when u is $O(1)$ and δ_T and δ_B are both very small,

$$\Delta(u) \sim \frac{\delta_T\,\delta_B}{\delta_T+\delta_B}\,u^3. \tag{5.1}$$

This approximation fails when u is large enough to make $u^3\delta_B$ of order unity, but if we assume that important contributions to the integral in (2.15) come from finite u, we can replace $\Delta(u)$ by its approximate form (5.1) in the first of the dual integral equations.

The resulting pair are of the type solved by Titchmarsh (1937, p. 337) and we find

$$A(u) = \frac{1}{8u}\left(\frac{1}{\delta_T}+\frac{1}{\delta_B}\right)J_3(u), \tag{5.2}$$

with a corresponding drag D given by

$$D = \frac{\pi}{48}\,\rho\Omega U a^3\left(\frac{1}{\delta_T}+\frac{1}{\delta_B}\right). \tag{5.3}$$

We can also work out the velocity field, and (as in I) denoting the interior of the upper Taylor column by T, the interior of the lower Taylor column by B and its exterior by E, we get

$$v_E = 0, \quad v_T = -\frac{U}{8\delta_T}\,s(1-s)^2, \quad v_B = \frac{U}{8\delta_B}\,s(1-s^2). \tag{5.4}$$

Now if the Ekman suction is to be negligible, we must have $vE^{\frac{1}{2}} \ll U$, so that (recalling $\delta_T \leqslant \delta_B$) we must have

$$E^{\frac{1}{2}}/\delta_T \ll 1.$$

This condition will be secured if $h \gg aE^{-\frac{1}{2}}$, provided that δ_T/δ_B is not small.

The velocity field (5.4) is not analytic across $r = a$ and satisfies instead the jump conditions

$$\left.\begin{aligned} &v \text{ continuous across } r = a,\\ &h_T\frac{\partial v_T}{\partial r}+h_B\frac{\partial v_B}{\partial r} = h\frac{\partial v_E}{\partial r} \quad (r=a). \end{aligned}\right\} \tag{5.5}$$

These are exactly the jump conditions at a Stewartson $E^{\frac{1}{3}}$ layer, as was shown in I by a generalization of Stewartson's (1966) argument. The $E^{\frac{1}{3}}$ layer is thin because $h \ll aE^{-1}$ while, because $h \gg aE^{-\frac{1}{2}}$, the $E^{\frac{1}{4}}$ layer occupies the whole of the interior of the Taylor column, a situation envisaged in I. In fact, the solution (5.4) can be obtained directly once the structure of the flow is recognized, but we shall not give the details.

The $E^{\frac{1}{3}}$ layer can be determined exactly as in I, the solution (5.4) yielding the constants $d_B^{(1)}$, $d_T^{(1)}$ and $d^{(1)}$ occurring in equations (5.35), (5.36) and (5.37) of I. The velocity in the shear layer is $O(E^{\frac{1}{3}})$ because the layer is merely removing a discontinuity in the derivative of the swirl velocity.

We shall not work out the details of the $E^{\frac{1}{3}}$ layer here, but we shall draw attention to the most important feature. The analysis of I shows that the swirl velocity in the shear layer tends to zero as the boundary-layer variable $(r-a)E^{-\frac{1}{3}}$ tends to plus infinity, but tends to non-zero values as $(r-a)E^{-\frac{1}{3}}$ tends to minus infinity, the values

δ	D/D_0 computed	D/D_0 from (4.8)	
10·0	1·00947	1·00946	
6·0	1·0156	1·0156	
4·0	1·0231	1·0231	
2·5	1·0364	1·0362	
1·5	1·0594	1·0589	
1·0	1·0873	1·0862	
0·6	1·1411	1·1383	
0·4	1·2060	1·2000†	
0·25	1·3183	1·3024†	
		D/D_0 from (5.7)	Difference $\times\, \delta^{\frac{1}{3}}$
0·006	9·144	7·573	0·286
0·004	12·440	10·699	0·276
0·0025	18·033	16·060	0·268
0·0015	27·410	25·137	0·260
0·001	38·593	36·042	0·255

TABLE 1. Comparison, in the case $\delta_T = \delta_B = \delta$, of D/D_0 computed using the truncation method with the approximate solutions (4.8), valid for $\delta \gg 1$, and (5.7), valid for $\delta \ll 1$. Shanks's transformation was used in the entries distinguished with †.

being different in T and B. This implies that an $O(E^{\frac{1}{3}})$ correction must be added to the solution (5.4); this correction can be shown to take the form of a rigid rotation.

The rotation rates are different in T and B, so that there is an $O(E^{\frac{1}{3}})$ correction to the drag formula (5.3). The calculation is straightforward and only the final result will be given, which is

$$D = \frac{\pi\rho U\Omega a^3}{48}\left(\frac{1}{\delta_T}+\frac{1}{\delta_B}\right)\left\{1-\frac{12\zeta(\frac{1}{3})}{(2\pi)^{\frac{1}{3}}}\left(\delta_T^{\frac{1}{3}}+\delta_B^{\frac{1}{3}}-(2\delta)^{\frac{1}{3}}\right)+\ldots\right\}. \tag{5.6}$$

This result enables us to find D at values of δ for which the truncation method fails. However it is practically useful only at very small values of δ. When $\delta_T = \delta_B$ we find that

$$\frac{D}{D_0} = \frac{\pi}{128\delta}\left(1+4{\cdot}685\ldots\delta^{\frac{1}{3}}+\ldots\right) \tag{5.7}$$

and the correction term is not small even when $\delta = 0{\cdot}001$. A comparison of (5.7) with the results obtained by the truncation method is given in § 6, which describes our results and compares them with Maxworthy's experiments.

6. Results and comparison with experiment

The infinite system of equations (3.8) was solved approximately by replacing it by an $N \times N$ system. Spherical Bessel functions of orders up to 29 were available in a library subroutine, so the largest value of N we could use was 14. To see whether this was large enough we compared the values of a_0 computed with $N = 5$ with those computed with $N = 14$. When $\delta_T = \delta_B = \delta$ we found no significant difference for $\delta = 10{\cdot}0$, $1{\cdot}0$ and $0{\cdot}1$ and only a difference of $0{\cdot}003$ % when $\delta = 0{\cdot}001$.

The comparison between the computed results and those obtained from the approximate formulae for $\delta \gg 1$ and $\delta \ll 1$ are shown in table 1. Evidently (4.8) is a good approximation even when δ is as small as $1{\cdot}0$. However, as we anticipated in § 5,

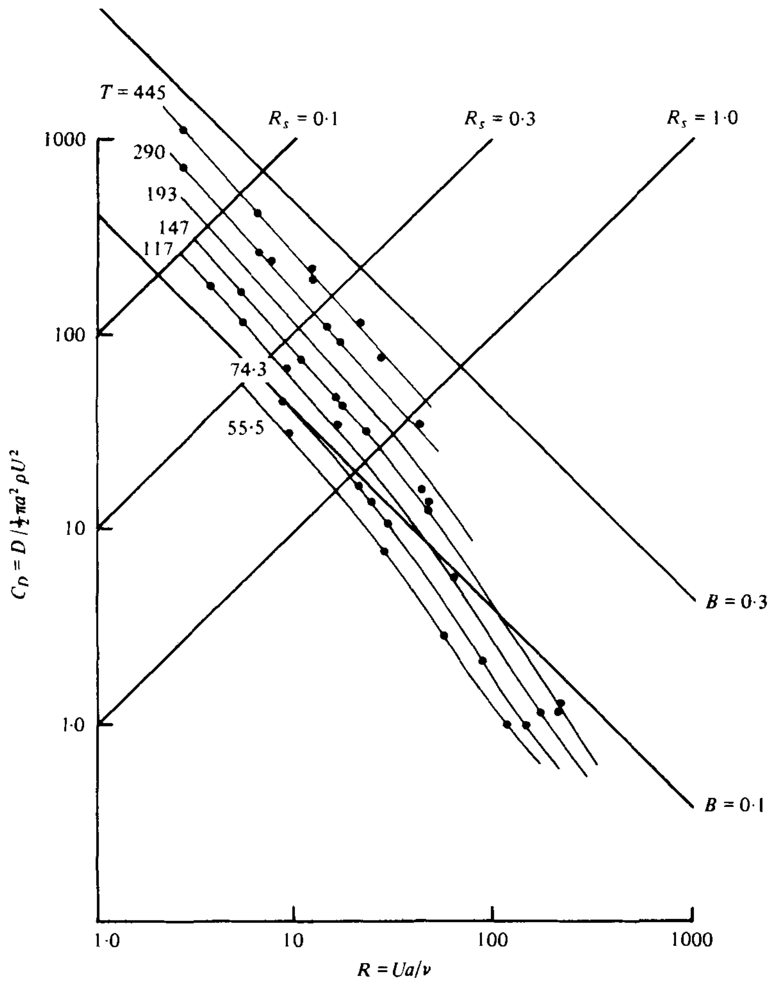

FIGURE 1. Estimated effects of Ekman suction and nonlinearity in the shear layers on Maxworthy's data.

(5.7) is not a good approximation even when $\delta = 0{\cdot}001$. The results are consistent with the next term in the expansion being proportional to $\delta^{-\frac{1}{3}}$, as is shown in the last column of the table.

Before we compare our predictions with Maxworthy's experiments we must obtain the conditions under which our theory is valid.

In the linear theory itself we have neglected the effect of Ekman suction and, as we saw in § 5, this requires $B \ll 1$, where

$$B = aE^{-\frac{1}{2}}/h. \tag{6.1}$$

The linear equations will be valid provided the appropriate Rossby number is small. In the Stewartson shear layer both velocities and velocity gradients are larger than in the rest of the flow. When $\delta = O(1)$ the velocities are $O(UE^{-\frac{1}{6}})$ in a layer of thickness $O(aE^{\frac{1}{3}})$, so that the local Rossby number is

$$R_S = R_0 E^{-\frac{1}{2}}. \tag{6.2}$$

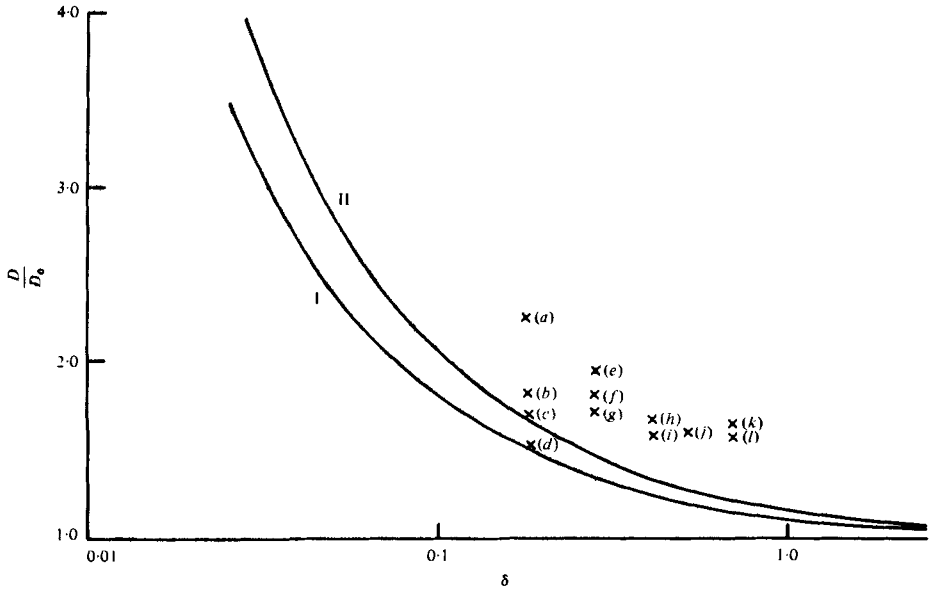

FIGURE 2. Comparison of theoretical (I, $\delta_T = \delta_B$; II, $\delta_T = \frac{1}{3}\delta_B$) drag ratio D/D_0 withMaxworthy's (1970) experiments. The values of R_S at the labelled points are (*a*) 0·06, (*b*) 0·10, (*c*) 0·15, (*d*) 0·29, (*e*) 0·08, (*f*) 0·22, (*g*) 0·19, (*h*) 0·25, (*i*) 0·27, (*j*) 0·23, (*k*) 0·18 and (*l*) 0·25. In each column of points the drag ratio decreases as R_S increases, with the exception of the value for the point (*f*).

Since the shear layer get thinner and the velocities larger near the equator of the sphere in the unbounded case, as we can see from I, nonlinear effects are likely to be stronger there. However, we shall ignore this complication at present and merely require $R_S \ll 1$. This condition is not changed when $\delta \ll 1$, provided $h \gg aE^{-\frac{1}{2}}$, as we have assumed throughout

We can now examine Maxworthy's data with these restrictions in mind. Maxworthy's results show that, roughly,

$$C_D = 2/R_0.$$

If we adopt this formula and the representative value $h/a = 100$, we can, with sufficient accuracy for the purpose of a survey, place the lines B = constant and R_S = constant on Maxworthy's plot of the drag coefficient C_D as a function of the Reynolds number $Ua\nu^{-1}$. The results are shown in figure 1. Evidently Ekman-suction effects are, on the present estimate, not large for his data whereas nonlinear effects can be expected to affect the bulk of his results.

We have used these estimates to select data to compare with our theory, the points chosen being those above the line $R_S = 0{\cdot}3$. (Professor Maxworthy kindly made available the original of his figure and we obtained the data from this.) Maxworthy measured transit times of his freely rising spheres, so that his velocities are averages. However, he has told us that $0{\cdot}43 < h_T/h_B < 1{\cdot}33$ in his measurements, so that, if our theory is right, the measured drag should lie between our predicted values for $h_T/h_B = 1$ and $h_T/h_B = 0{\cdot}43$.

In figure 2 we show a comparison of our theory and Maxworthy's data. The two curves show the dependence of the drag ratio D/D_0 on δ for

$$h_T/h_B = 1 \quad \text{and} \quad h_T/h_B = 0{\cdot}33.$$

The effect of position is not large in the range of interest.

The agreement is poor and we believe that the end-wall effect is responsible for only part of the discrepancy between Stewartson's prediction for an unbounded region and experiment.

We stress that our theory fails in two respects.

The drag ratio inferred from Maxworthy's data is not just a function of δ and h_T/h_B, because the spread of points at given δ is too large to be accounted for by position effects. Nor is the spread due to experimental scatter. At any δ the drag ratio D/D_0 decreases with increasing R_S, suggesting that nonlinear effects are present.

The drag ratio at the smallest R_S is systematically higher than that predicted by our theory. The reason for this discrepancy is not known. It is clear from figure 2 that the discrepancy is becoming constant as δ increases, i.e. as the end walls recede, which suggests that at the same Ekman number a similar discrepancy would be present in unbounded flow.

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